

The zeroth law in quasi-homogeneous thermodynamics and black holes

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In this work we consider the zeroth law of thermodynamics for systems whose entropy is a quasi-homogeneous function of the extensive variables. We show that the generalized Gibbs-Duhem identity and the Maxwell construction for phase coexistence based on the standard zeroth law are incompatible in this case. Besides, we argue that the generalized Gibbs-Duhem identity suggests a revision of the zeroth law which in turns permits to reconsider Maxwell's construction in analogy with the standard case. The physical feasibility of our proposal as well as its relationship with previously formulated generalizations of the zeroth law in the context of non-extensive statistical mechanics are considered in the particular case of black holes. Further possible applications are also discussed.

I. INTRODUCTION

One of the central problems in modern theoretical physics is that of understanding the thermodynamic properties of systems with long-range interactions. For such systems the standard (additive) Boltzmann-Gibbs (BG) entropy diverges in the thermodynamic limit, due to the divergence of the interaction part of the Hamiltonian [1]. Therefore, different models generalizing the BG entropy have been proposed in the literature [1], with quite encouraging results reestablishing the thermodynamic limit, hence enabling a sound thermodynamic treatment.

However, such treatments yield macroscopic entropies that are not in general homogeneous functions of degree one of the extensive variables, but have a generalized “quasi-homogeneous” behavior (see e.g. [1–3]). For this reason it is of interest to investigate the laws of thermodynamics in the case of quasi-homogeneous systems, i.e., systems for which the entropy is a generalized quasi-homogeneous function, with possibly different degrees of homogeneity for each of its variables [4, 5].

The problem of understanding equilibrium for systems with long-range interactions has been addressed in the literature from the statistical mechanical perspective in the last years. The results point to the fact that for systems with long-range interactions, the standard thermodynamic temperature is not the correct parameter to be equated at thermal equilibrium [5–11]. This is not unexpected, since it has been known in general relativity for almost a century that the standard thermodynamic

temperature is not a correct measure of thermal equilibrium in the description of self-gravitating gases [5, 12–14]. Further evidence comes from recent numerical and experimental verifications, see e.g. [8, 9, 15].

In this work we address the redefinition of the zeroth law from a macroscopic perspective. We consider systems whose entropy is a quasi-homogeneous function of the extensive variables and find a zeroth law which stems from a generalized version of the Gibbs-Duhem identity for such systems, rooted in the formalism for quasi-homogeneous thermodynamics developed in [2, 3]. We show that such revision is essential if one wants to deal with Maxwell's construction for phase coexistence (cf. [16] and the discussion in Section 4.3 in [17]). In fact, Maxwell's equal area law based on the standard zeroth law of thermodynamics leads to a mathematical inconsistency with the generalized Gibbs-Duhem identity in the case of systems with quasi-homogeneous entropy. However, using our prescription, we can remedy this inconsistency – at the price of introducing a slight modification of the variables defining equilibrium. Our result provides a direct prediction of the new equilibrium quantities and thus can be tested through numerical as well as laboratory experiments.

A relevant application of our framework concerns black hole thermodynamics, where the thermodynamic variables scale differently when rescaling the system, according to the Smarr relation. In these cases, the entropy is a quasi-homogeneous function of the thermodynamic variables describing the system [4]. Several authors have established the existence of first order phase transitions – i.e., coexistence processes – based on the analogy with the van der Waals phase diagram, using the Maxwell equal area law to find the coexistence curve (see e.g. [4, 18–21] and references therein). However, it is usually recognized that using standard thermodynamics in the case of black holes is not fully justified and that an extension of the thermodynamic laws for systems with quasi-

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homogeneous entropy is called for [4, 5]. With our revision of the zeroth law in this work we take a step forward in this direction and obtain a modified notion of equilibrium, which in the examples of the black holes we consider leads us to two important consequences: first, we show that for the Schwarzschild black hole our new temperature coincides (up to a constant factor) with the result in [22], where such parameter is obtained using the generalized zeroth law for non-extensive statistical mechanics developed in [10]. This shows that, at least in the Schwarzschild case, there is a consistency between different approaches, although it remains to be seen whether this consistency persists in other cases such as Kerr. Secondly, we consider the first order phase transition in the Kerr–Anti de Sitter (Kerr–AdS) family of black holes and show that using the new equilibrium parameters derived according to our prescription such transition seems to disappear. Given the importance of the example in the context of the AdS/CFT correspondence, we believe that this can be relevant for future investigation.

This paper is structured as follows. In Section II we point out the aforementioned mathematical problem with the use of Maxwell’s construction based on the standard zeroth law of thermodynamics and the Gibbs-Duhem relation in the case of quasi-homogeneous thermodynamic potentials. We continue in Section III by proposing a generalized form of the zeroth law, which is consistent with the corresponding generalized Gibbs-Duhem relation. To illustrate the new form of the zeroth law, we consider the examples of Schwarzschild and Kerr–Anti de Sitter black holes in Section IV, before we conclude in Section V. Throughout this work we use Planck units, in which $c = G = \hbar = k_B = 1$.

II. A MATHEMATICAL INCONSISTENCY

In this section we introduce the problem of the mathematical inconsistency between the usual zeroth law of thermodynamics, the standard Maxwell construction for coexistence between different phases and the generalized Gibbs-Duhem identity.

We begin by considering the ordinary case in which the entropy is a homogeneous function of degree one of the extensive variables, i.e.,

$$S(\lambda U, \lambda V, \lambda N) = \lambda S(U, V, N). \quad (1)$$

Here one can easily prove that the first derivatives of S with respect to the extensive quantities are homogeneous functions of order zero and therefore they do not scale with the size of the system. This is the reason why they are usually referred to as *intensive* quantities. Furthermore, using Euler’s theorem for homogeneous functions of order one,

$$S = \frac{1}{T}U - \frac{p}{T}V + \frac{\mu}{T}N, \quad (2)$$

and the first law of thermodynamics

$$dS = \frac{1}{T}dU - \frac{p}{T}dV + \frac{\mu}{T}dN, \quad (3)$$

one arrives at the Gibbs-Duhem relation

$$Ud\left(\frac{1}{T}\right) - Vd\left(\frac{p}{T}\right) + Nd\left(\frac{\mu}{T}\right) = 0, \quad (4)$$

which is a mathematical identity stating that the intensive quantities are not all independent [16].

Note the important fact that the Gibbs-Duhem identity (4) is mathematically consistent with Maxwell’s law for phase coexistence based on the standard zeroth law: here one considers a single system splitting into two different phases remaining at equilibrium (i.e., same values of the intensive quantities between the two phases), while the entropy and volume of the system change (causing a discontinuity in the extensive quantities, and thus giving rise to the name *first order phase transition*). This is clearly consistent with (4).

In what follows we show that such consistency is lost if one applies the standard zeroth law together with the standard Maxwell construction to thermodynamic systems whose entropy is not a homogeneous function of degree one with respect to the extensive variables.

Let us start by recalling some definitions.

Definition II.1 (Quasi-homogeneous function). *A function G of a set of variables $\mathbf{x} = (x_1, \dots, x_n)$ is quasi-homogeneous of degree β and type $(\beta_1, \dots, \beta_n)$ if [2]*

$$G(\lambda^{\beta_1}x_1, \dots, \lambda^{\beta_n}x_n) = \lambda^\beta G(x_1, \dots, x_n). \quad (5)$$

Homogeneous functions are a particular case of quasi-homogeneous functions, recovered by setting all $\beta_a = 1$, $a = 1, \dots, n$. From this point onwards, we will use a notation consistent with [23]. Therefore, let us denote the extensive variables by q^a and use p_a to denote their conjugate variables, i.e., the first derivatives of the entropy with respect to the q^a .

Let us now proceed to consider non-standard cases, in which $S(q^a)$ is a quasi-homogeneous function of degree β as in (5). Then its partial derivatives with respect to the extensive variables are quasi-homogeneous functions of degree $\beta - \beta_a$, that is,

$$p_a(q^b) \equiv \frac{\partial}{\partial q^a} S(q^b), \quad (6)$$

with

$$p_a(\lambda^{\beta_1}q^1, \dots, \lambda^{\beta_n}q^n) = \lambda^{\beta - \beta_a} p_a(q^b). \quad (7)$$

It is clear that in such case the conjugated variables p_a are not intensive, i.e., their values do not remain the same when the system is rescaled. This provides us with a first indication of the fact that they are not quite suitable parameters to define thermodynamic equilibrium. An alternative argument goes as follows.

Given a system with quasi-homogeneous entropy, one can use Euler's theorem for quasi-homogeneous functions

$$\beta S(q^a) = \sum_{a=1}^n \beta_a p_a q^a \quad (8)$$

and the first law

$$dS = \sum_{a=1}^n p_a dq^a \quad (9)$$

to work out the *generalized Gibbs-Duhem relation* [2]

$$\sum_{a=1}^n \beta_a \left[\left(1 - \frac{\beta}{\beta_a} \right) p_a dq^a + q^a dp_a \right] = 0. \quad (10)$$

In the particular case of S being homogeneous of degree β , the above formula reduces to

$$(1 - \beta) \sum_{a=1}^n p_a dq^a + \sum_{a=1}^n q^a dp_a = 0. \quad (11)$$

Equation (10) (resp. (11)) is thus the generalization of (4). It is a mathematical identity that must be satisfied along any process at equilibrium, provided the entropy is quasi-homogeneous (resp. homogeneous) of degree β . Notice also that (10) reduces to the standard case whenever the entropy is a homogeneous function of degree one. If S is not homogeneous of degree one, the first term inside the square brackets in (10) must be different from zero.

From the above discussion on the role of the zeroth law in Maxwell's construction and its consistency with the ordinary Gibbs-Duhem relation, it should be evident why such consistency is lost in the case of the generalized Gibbs-Duhem identity (10). Indeed for the two phases to be at equilibrium, the standard zeroth law would predict that no change in any of the p_a should happen, i.e., $dp_a = 0$ for all a . This implies that the second term in (10) vanishes identically. However, in general the first term in (10) is different from zero, thus providing an inconsistency. For instance, in the case of a homogeneous entropy of degree β , it follows from equations (9) that the first term of (11) is proportional to the jump of the entropy during the transition, and hence to the latent heat. This inconsistency leads to the two following possibilities: either one gives up the standard formulation of phase coexistence expressed by the Maxwell construction (at least in its usual form) or one has to change the conditions for equilibrium, i.e., the zeroth law. Due to the many indications arising from different perspectives pointing to the fact that the zeroth law needs to be revisited for such systems (see e.g. [5–11]), we opt for the latter route. In the next section we show that the Gibbs-Duhem identity can be used to derive new quantities which are truly intensive quantities under certain rescaling of the extensive variables, and thus should be considered to be the appropriate parameters to describe the state of equilibrium for systems with quasi-homogeneous entropy.

III. REVISITING THE ZEROTH LAW OF THERMODYNAMICS

In this section we present a revision of the definition of thermodynamic equilibrium for systems whose entropy is a quasi-homogeneous function of any degree in a way that is always consistent with the generalized Gibbs-Duhem identity (10).

From the analysis of the homogeneity of the first derivatives of the entropy – see (7) – let us propose the following

Definition III.1 (Generalized intensive variables). *Let $S = S(q^a)$ be a quasi-homogeneous function of degree β of a set of n extensive variables $\{q^a\}$. Let p_a denote the partial derivatives of S as in (6). The quantities*

$$P_a(q^b) \equiv \frac{p_a(q^b)}{(q^a)^{\beta/\beta_a - 1}}. \quad (12)$$

are called the generalized intensive variables.

Note that these quantities are *quasi-homogeneous of degree zero* under a rescaling of the extensive variables as defined in (5), which means that their values do not change under such rescaling of the system. This is a desirable property for quantities defining a measure of equilibrium (note that they play a central role already in the formalism developed in [2], although in this work they were not advocated as the quantities responsible for equilibrium).

As a particular case, whenever the entropy is homogeneous of degree β , the generalized intensive variables reduce to

$$P_a(q^b) \equiv \frac{p_a(q^b)}{(q^a)^{\beta - 1}}. \quad (13)$$

In this case, since the extensive variables q^a scale with the size of the system, and the $p_a(q^b)$ behave according to (7), it is straightforward to verify that the $P_a(q^b)$ are homogeneous functions of degree zero.

Definition III.1 suggests the following modification of the notion of thermodynamic equilibrium:

Definition III.2 (Thermodynamic Equilibrium). *Two systems whose entropy is a quasi-homogeneous function of the same degree are in thermodynamic equilibrium with each other if and only if they have the same values of the $P_a(q^b)$.*

This is the *generalized zeroth law of thermodynamics* that we propose for any system whose entropy is a quasi-homogeneous function of the extensive variables.

Now let us show that Definition III.2 is mathematically consistent with the generalized Gibbs-Duhem identity (10), even when considering processes of coexistence as in the case of the usual Maxwell equal area law. Consider a system whose thermodynamic entropy is quasi-homogeneous and a process during which the system

evolves at fixed values of the equilibrium quantities as defined in (12). From equation (12) we have

$$p_a(q^b) = P_a(q^b) (q^a)^{\beta/\beta_a-1}, \quad (14)$$

and we can thus rewrite the generalized Gibbs-Duhem identity (10) in terms of the $P_a(q^b)$ as

$$\sum_{a=1}^n \beta_a \left[\left(1 - \frac{\beta}{\beta_a}\right) P_a(q^a)^{\beta/\beta_a-1} dq^a + q^a d \left(P_a(q^a)^{\beta/\beta_a-1} \right) \right] = 0. \quad (15)$$

By explicit calculation of the second term, we can rewrite the above identity as

$$\begin{aligned} 0 &= \sum_{a=1}^n \beta_a \left[\left(1 - \frac{\beta}{\beta_a}\right) P_a(q^a)^{\beta/\beta_a-1} dq^a + q^a \left[(q^a)^{\beta/\beta_a-1} dP_a + \left(\frac{\beta}{\beta_a} - 1\right) (q^a)^{\beta/\beta_a-2} P_a dq^a \right] \right] \\ &= \sum_{a=1}^n \beta_a (q^a)^{\beta/\beta_a} dP_a. \end{aligned} \quad (16)$$

Finally, considering the above generalized zeroth law of thermodynamics, which establishes that the generalized intensive variables remain constant, $dP_a = 0$, it is easy to check that the generalized Gibbs-Duhem identity is satisfied. This shows that our new definition of thermodynamic equilibrium (Definition III.2), based on equating the true generalized intensive quantities (Definition III.1), is mathematically consistent with the generalized Gibbs-Duhem identity.

We remark that with our prescription one can consider the example of a process of coexistence among different phases at equilibrium without any incongruence, as long as the definition of equilibrium is given by equating the quantities in (12). Note also that our simple redefinition gives a general prediction about the quantities that have to be constant at equilibrium.

In the next section we consider examples from black holes thermodynamics and show that for the Schwarzschild black hole our redefinition of the equilibrium condition coincides with a different instance of the generalized zeroth law of thermodynamics resulting from non-extensive statistical mechanics [22], while for the Kerr-Anti de Sitter black hole our construction suggests that a reconsideration of the first order phase transition might be in order.

IV. EXAMPLES

In this section we investigate some examples for the above ideas in the context of black hole thermodynamics. In principle, our generalization of the zeroth law

can be applied to any black hole system, given that one can easily determine the degrees of homogeneity from the Smarr relation,

$$(D-3)M = (D-2)TS + (D-2)\Omega J - 2PV + (D-3)\Phi Q, \quad (17)$$

where D is the number of spacetime dimensions, M is the mass of the black hole, T is the Hawking temperature, S is the Bekenstein-Hawking entropy and the other terms are work terms depending on the black hole family in question [4]. Here, we consider two in particular, namely the Schwarzschild and the Kerr-AdS black holes, to compare our results with previous proposals and to illustrate new features.

A. Schwarzschild

The Schwarzschild black hole is the most straightforward example, since its thermodynamics is described by only one extensive variable, i.e., its mass M . The entropy as a function of M is

$$S(M) = 4\pi M^2, \quad (18)$$

which is a homogeneous function of degree $\beta = 2$. From this the standard temperature is derived as

$$\frac{1}{T} = \frac{\partial S}{\partial M} = 8\pi M. \quad (19)$$

It is immediate to see that this a homogeneous function of degree 1 with respect to M , the parameter defining the scale of the system, and therefore is not a real intensive quantity. With (19) and using (13), we can obtain the generalized temperature as

$$\tilde{T} = TM = \frac{1}{8\pi}, \quad (20)$$

i.e., a constant. Note that, a constant is – trivially – a real intensive quantity, as it does not change with any scaling of M .

A similar result, i.e., a constant generalized temperature, has been obtained previously for the Schwarzschild black hole [22] by using the generalized zeroth law derived from non-extensive statistical mechanics proposed in [10]. In this work, the most general conditions for thermal equilibrium of systems with non-additive energy and entropy are established by using a method based on the definition of the formal logarithms of these quantities. However, the same method was also applied in [24] in the analysis of the Kerr black hole, resulting in a constant generalized temperature, regardless of the angular momentum, identical to the Schwarzschild case – an indication that the result may be unphysical, as the authors point out themselves. Moreover, from our formalism a dependence of the generalized temperature on the angular momentum is to be expected. Finally, in [25, 26] the

Rényi entropy was used as the formal logarithm of the Bekenstein-Hawking entropy. In this case the temperature for the Schwarzschild case depends on the mass M and is not intensive. The connection of our proposal to these approaches and the general question of the underlying behaviour of the energy and entropy is thus not quite clear and might be addressed in future works.

B. Kerr-AdS

Kerr black holes in asymptotically Anti de Sitter space are thermodynamically determined by three extensive variables, namely their mass M , angular momentum J and pressure P , which is defined via the cosmological constant Λ of the spacetime as

$$P = -\frac{\Lambda}{8\pi G}. \quad (21)$$

The cosmological constant is usually included as a pressure into the thermodynamic description of black holes [4, 18, 27], and thus it turns out that the internal energy of the black hole is

$$U = M - PV, \quad (22)$$

and therefore the mass of the black hole is identified with the enthalpy

$$M \equiv H = U + PV. \quad (23)$$

For the Kerr-AdS black hole one obtains [18]

$$H(S, P, J) = \frac{1}{2} \sqrt{\frac{4\pi^2 J^2 \left(\frac{8PS}{3} + 1\right) + \left(\frac{8PS^2}{3} + S\right)^2}{\pi S}}, \quad (24)$$

and from this, provided $J \neq 0$, it is possible to calculate the expression for the internal energy as

$$U(S, V, J) = \left(\frac{\pi}{S}\right)^3 \left[\left(\frac{3V}{4\pi}\right) \left\{ \frac{S^2}{2\pi^2} + J^2 \right\} - J^2 \left\{ \left(\frac{3V}{4\pi}\right)^2 - \left(\frac{S}{\pi}\right)^3 \right\}^{1/2} \right]. \quad (25)$$

For simplicity and without loss of generality, we will limit further analyses to positive angular momenta, i.e., $J > 0$. The temperature and pressure can be easily obtained as

$$T = -\frac{1}{8S^4} \left[\frac{6\pi^{3/2} J^2 (8S^3 - 9\pi V^2)}{\sqrt{9\pi V^2 - 16S^3}} + 18\pi^2 J^2 V + 3S^2 V \right], \quad (26)$$

$$P = \frac{3}{8S^3} \left[2\pi^2 J \left(J - \frac{3\sqrt{\pi} JV}{\sqrt{9\pi V^2 - 16S^3}} \right) + S^2 \right]. \quad (27)$$

The case of Kerr-AdS is particularly interesting for our purposes because its equation of state, i.e., the relation $P(V, T)$ at fixed J , qualitatively shows the same behaviour as a van der Waals fluid, which is generally taken as an indication of the presence of a first order phase transition, sometimes referred to as the CCK phase transition [20]. To see this let us fix $J = 1$ from now on and first look at Fig. 1, where we plot $P(V, T)$ as a function of V for various choices of T , with the inset zooming in on one of the curves to show the characteristic van der Waals bump. The region of the bump is the area where one would apply the Maxwell equal area law in analogy to ordinary thermodynamics [18]. A different (equivalent) way to look at such transition is by considering the graph of the Gibbs free energy,

$$G(T, P, J) = U - TS - PV. \quad (28)$$

We show the plot of G as a function of T and P in Fig. 2, which clearly displays the characteristic swallowtail region corresponding to a first order phase transition. To illustrate the multi-valued behaviour of the Gibbs free en-

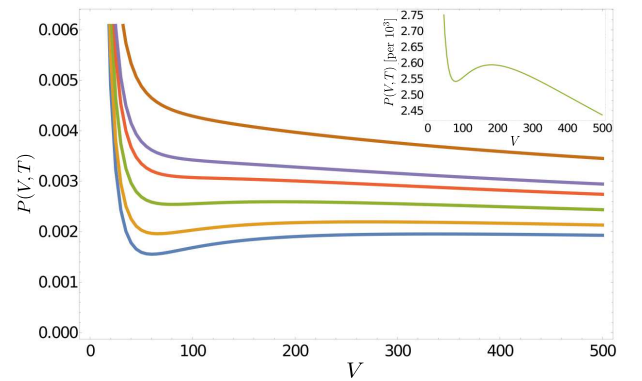


FIG. 1: Equation of state $P(V, T)$ for different values of T and with $J = 1$.

ergy more in detail, in Figs. 3 and 4 we show cuts along lines of constant T and P , again featuring the characteristic swallowtails.

Now we consider the analysis in terms of the gener-

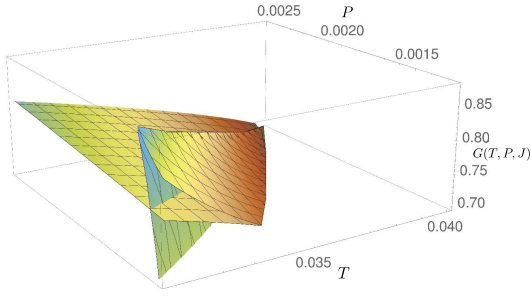


FIG. 2: Gibbs free energy $G(T, P)$ as a function of T and P at $J = 1$, showing the characteristic swallowtail.

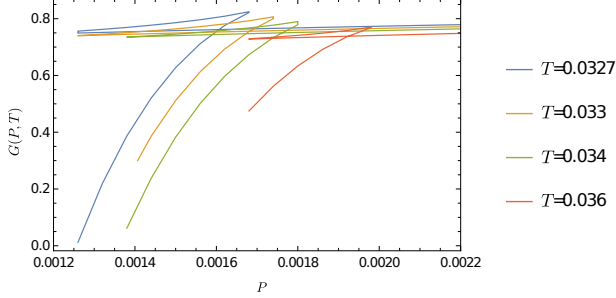


FIG. 3: Cuts of the Gibbs free energy at constant T

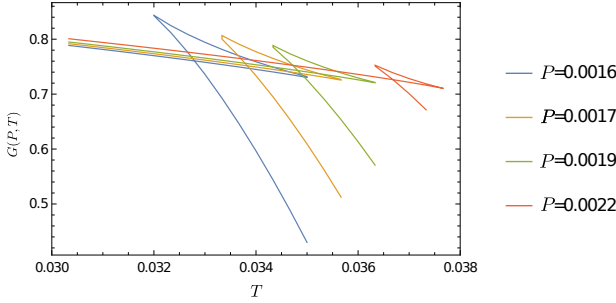


FIG. 4: Cuts of the Gibbs free energy at constant P .

alized intensive quantities defined in (12). The generalization of the thermodynamic intensities in the case of Kerr-AdS is straightforward: using the Smarr relation (17) applied to four spacetime dimensions, we have

$$S = \frac{1}{2T}U + \frac{3}{2}\frac{P}{T}V - \frac{\Omega}{T}J. \quad (29)$$

From this we can determine the degrees of homogeneity of the variables T and P as $\beta_T = 1/2$ and $\beta_P = 3/2$ respectively (we will not need to consider the generalized variable corresponding to the angular velocity here). The overall degree of homogeneity of the entropy is $\beta = 1$. From Eq. (12) we can then infer the generalized variables responsible for equilibrium from

$$\frac{1}{\tilde{T}} = \frac{1}{TU} \quad \text{and} \quad \frac{\tilde{P}}{\tilde{T}} = \frac{P}{T}V^{1/3}. \quad (30)$$

Combining the two expressions, we end up with the generalized thermodynamic equilibrium parameters

$$\tilde{T} = TU \quad \text{and} \quad \tilde{P} = PUV^{1/3}. \quad (31)$$

By construction, these functions are quasi-homogeneous of degree 0 with respect to the correspondingly rescaled extensive variables, i.e.,

$$\tilde{T}(\lambda^1 S, \lambda^{3/2} V, \lambda^1 J) = \lambda^0 \tilde{T}(S, V, J), \quad (32)$$

$$\tilde{P}(\lambda^1 S, \lambda^{3/2} V, \lambda^1 J) = \lambda^0 \tilde{P}(S, V, J). \quad (33)$$

In terms of S and V (for $J = 1$) these read

$$\begin{aligned} \tilde{T}(S, V) = & \frac{3V\pi^{13/2}}{8S^7} \left[-36 \left(\frac{S}{\pi} \right)^3 - 10 \left(\frac{S}{\pi} \right)^5 + 27 \left(\frac{V}{\pi} \right)^2 + 9 \left(\frac{S}{\pi} \right)^2 \left(\frac{V}{\pi} \right)^2 \right] \\ & + \frac{3\pi^6}{64S^7} \sqrt{9\pi V^2 - 16S^3} \left[32 \left(\frac{S}{\pi} \right)^3 - 72 \left(\frac{V}{\pi} \right)^2 - 24 \left(\frac{S}{\pi} \right)^2 \left(\frac{V}{\pi} \right)^2 - 3 \left(\frac{S}{\pi} \right)^4 \left(\frac{V}{\pi} \right)^2 \right], \end{aligned} \quad (34)$$

$$\tilde{P}(S, V) = \frac{9\pi^{5/2}V^{4/3}}{32S^6} \sqrt{9\pi V^2 - 16S^3} (2\pi^2 + S^2) \left(6\pi^2 V + 3S^2 V - 2\pi^{3/2} \sqrt{9\pi V^2 - 16S^3} \right). \quad (35)$$

Note that in order to show the quasi-homogeneity of these functions by rescaling the extensive variables, it is necessary to recover the terms containing J , including it as an extensive variable. Using these expressions, we can return to the plot of the equation of state, but now in

terms of the new variables, plotting $\tilde{P}(V, \tilde{T})$ as a function of V for different choices of \tilde{T} . As can be seen in Fig. 5, the curves are monotonously decreasing, therefore the system appears to be stable and there is no necessity for the Maxwell construction. The same effect can be

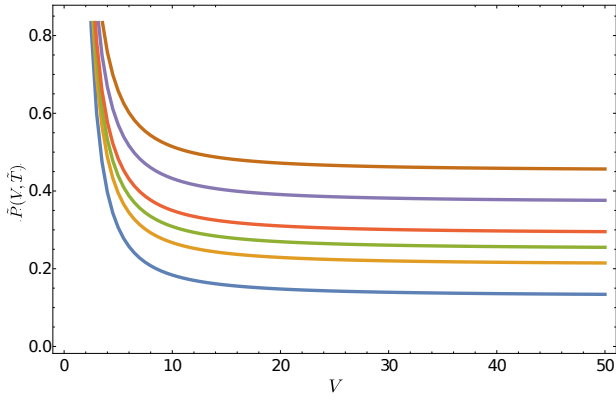


FIG. 5: Equation of state $\tilde{P}(V, \tilde{T})$ at constant \tilde{T} .

observed using the Gibbs free energy. We can re-express definition (28) in terms of the new intensive variables \tilde{T} and \tilde{P} and calculate the function $G(\tilde{T}, \tilde{P}, J)$, inverting Eqs. (34) and (35) numerically. The result is shown in Fig. 6, where the range of the function in the (\tilde{T}, \tilde{P}) -plane is limited by the allowed range of values for T and P , which in turn is determined by the positivity of the root occurring in Eqs. (26) and (27).

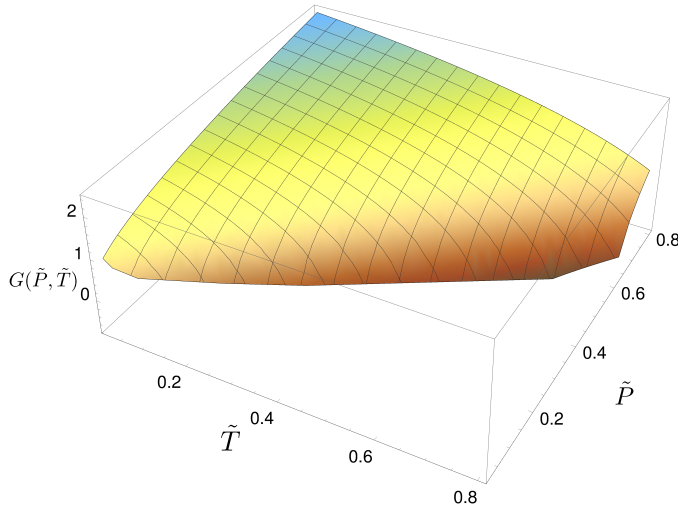


FIG. 6: Gibbs free energy $G(\tilde{T}, \tilde{P})$ as a function of \tilde{T} and \tilde{P} at $J = 1$, showing no swallowtail.

We see that the Gibbs free energy in terms of the generalized intensities is a single-valued smooth surface and shows no swallowtail. The same smooth behaviour is found in the cuts of G along constant \tilde{T} and \tilde{P} , as plotted in Figs. 7 and 8.

We conclude that for the Kerr–AdS black hole the use of the generalized intensive variables in the generalized zeroth law of thermodynamics proposed here as the parameters defining thermodynamic equilibrium seems to indicate that there is no first order phase transition between large and small black holes, as previously argued in the literature. However, our results deserve more in-

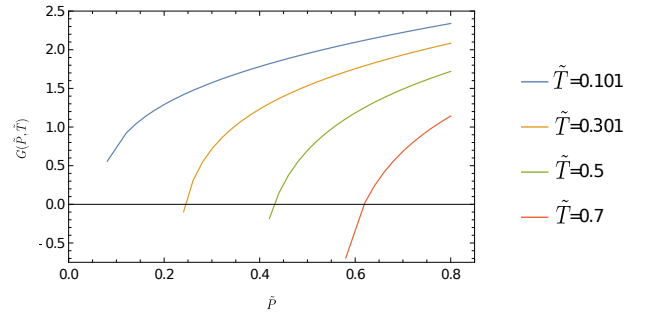


FIG. 7: Cuts of the Gibbs free energy at constant \tilde{T} .

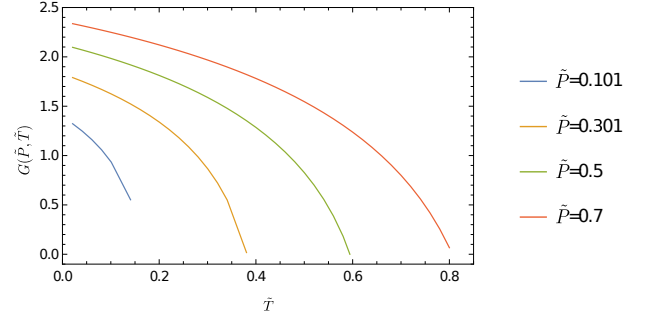


FIG. 8: Cuts of the Gibbs free energy at constant \tilde{P} .

vestigation. Perhaps a comparison with explicit models directly constructed from statistical mechanics could shed more light on the validity of such statements. Alternatively, an analysis involving thermodynamic response functions could be interesting, although the significance of these response functions in the context of a generalized zeroth law should be re-evaluated.

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we consider a generalization of the zeroth law of thermodynamics for systems whose thermodynamic entropy is a quasi-homogeneous function of the extensive variables (Definition III.2). Originating from the generalized version of the Gibbs-Duhem identity, we show how to define the generalized intensive variables that can be used to define thermodynamic equilibrium in such general cases (Definition III.1). Moreover, we prove that this new definition resolves an inconsistency between the use of the standard Maxwell equal area law and the generalized Gibbs-Duhem identity that is usually overlooked, especially in the literature regarding the thermodynamics of black holes. Within this context, we consider two examples where the application of our generalized zeroth law should be relevant, namely the Schwarzschild and the Kerr–AdS black holes. The former is important because with our approach we recover a previous result found in [22], derived from a different perspective. The latter example is of interest because in

the usual treatment the Kerr–AdS family of black holes shows a behavior which is very similar to that of a van der Waals fluid, including a first order phase transition. However, we argue that using the generalized intensive variables that we have introduced here in order to define thermodynamic equilibrium, such phase transition disappears. This statement however should be further investigated in other contexts in order to corroborate such a conclusion.

Our results are intended to be a step forward towards a deeper formal understanding of the thermodynamic properties of systems with quasi-homogeneous entropy. However, it also calls for more detailed investigations. In particular, we would like to extend the comparison between our approach and the one presented in [10, 22, 24–26] to other cases to see whether the agreement we found for the Schwarzschild black hole holds in more general

contexts. Besides, it would be worth using explicit calculations as in [8, 9] to check whether our prediction of the new thermodynamic parameters defining equilibrium can be tested by numerical experiments, and to compare our results with the formalism proposed in [28, 29] presenting a different instance of a generalized Gibbs–Duhem relation for systems with long-range interactions. These directions will be the subject of future work.

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